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Racah coefficients of the Lie superalgebra OSP(1, 2)

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Abstract. Using the indefinite metric, we study the coupling of three irreps of the OSP(1, 2) algebra, and give the general definition, orthogonality conditions, symmetry properties and calculating formulae of the Racah coefficients of the OSP(1, 2) algebra. Eight kinds of Racah coefficients exist for the OSP(1, 2) algebra and they are all expressed in terms of the Racah coefficients of the SO(3) algebra.

1. Introduction

The simpler Lie superalgebra OSP(1, 2) has been studied for ten years. Pais and Rittenberg [1] have given the irrep of the OSP(1, 2) algebra. Using a graded adjoint operation, Scheunert *et al* [2] have established the relation between the metrics of two isospin subspaces in an irrep of the OSP(1, 2) algebra and calculated the CG coefficients of the OSP(1, 2) algebra.

Our aim is to study the coupling of three irreps of the Lie superalgebra OSP(1, 2) and its Racah coefficients. The characteristics in this paper are as follows: (i) we use a new form of the irrep of OSP(1, 2), in which the basis vectors of the irrep J are labelled by $|2J, M\rangle$, where $M=-2J, -2J+1, \ldots, 2J$; (ii) we also use the graded adjoint operation. In view of the fact that the metrics of the coupling spaces are always indefinite, even if the metrics of the in-coupling spaces are positive definite, we adopt the indefinite metric to set the general theory on coupling from the very start.

The contents of this paper are arranged as follows: we describe the new form of the irrep of OSP(1, 2) in § 2, give the coupling theory with an indefinite metric in § 3 and study the coupling of three irreps of OSP(1, 2) and its Racah coefficients in § 4.

2. New form of irrep for OSP(1, 2)

We describe the new form of the irrep of OSP(1, 2). First, we write the OSP(1, 2) algebra as

$$
\begin{aligned}
&[Q_3, Q_\pm]=\pm 2Q_\pm \qquad [Q_+, Q_-]=Q_3 \\
&[Q_3, V_\pm]=\pm V_\pm \qquad [Q_\pm, V_\mp]=V_\pm \qquad [Q_\pm, V_\pm]=0 \qquad (2.1)\\
&\{V_\pm, V_\mp\}=-\tfrac{1}{4}Q_3 \qquad \{V_\pm, V_\pm\}=\pm\tfrac{1}{2}Q_\pm .
\end{aligned}
$$

We may show from these relations that the actions of generators on the basis vectors are as follows:

$$Q_3|2J, M\rangle = M|2J, M\rangle$$
$$Q_\pm|2J, M\rangle = A_\pm(M)|2J, M\pm 2\rangle \tag{2.2}$$
$$V_\pm|2J, M\rangle = B_\pm(M)|2J, M\pm 1\rangle$$

where (with a convenient choice of phases)

$$A_\pm(M) = \begin{cases} \frac{1}{2}[(2J\mp M)(2J\pm M+2)]^{1/2} & M\in\{2J, 2J-2, \ldots\} \\ \frac{1}{2}[(2J-M\mp 1)(2J+M\pm 1)]^{1/2} & M\in\{2J-1, 2J-3, \ldots\} \end{cases} \tag{2.3}$$

$$B_\pm(M) = \begin{cases} \mp[\frac{1}{8}(2J\mp M)]^{1/2} & M\in\{2J, 2J-2, \ldots\} \\ -[\frac{1}{8}(2J\pm M+1)]^{1/2} & M\in\{2J-1, 2J-3, \ldots\}. \end{cases} \tag{2.4}$$

Since the OSP(1, 2) algebra has been written in the form (2, 1), the Casimir operator is

$$K = Q_3^2 + 2(Q_+Q_- + Q_-Q_+) + 4(V_+V_- - V_-V_+) \tag{2.5}$$

and its eigenvalue is $2J(2J+1)$.

The basis vectors with $M\in\{2J, 2J-2, \ldots\}$ and those with $M\in\{2J-1, 2J-3, \ldots\}$ form two subspaces of the irrep J. We denote the degree of the basis vector $|2J, M\rangle$ by $\lambda(M)$ which has the same value for all basis vectors in a subspace, $\lambda = 0, 1$. We may show by means of the graded adjoint operation [2] that there is a relation between the metrics of two subspaces, namely

$$\varepsilon(2J, M)\varepsilon(2J, M+1) = a(-1)^{\lambda(2J)} \tag{2.6}$$

where $\varepsilon(2J, M) = \langle 2J, M|2J, M\rangle = \pm 1$, $\lambda(2J)$ is the degree of the subspace formed by the basis vectors with $M\in\{2J, 2J-2, \ldots\}$, while $a = \pm 1$ corresponds to the graded adjoint operations $V_-^\ddagger = aV_+$, $V_+^\ddagger = -aV_-$.

3. Coupling theory with indefinite metric

We first show the necessity of introducing the indefinite metric for the study of Racah coefficients, and then simply describe our theory on coupling.

3.1. Introduction of the indefinite metric

We denote the in-coupling spaces by J_1J_2 and the coupling space by J. In our new form $2J = 2J_1+2J_2, 2J_1+2J_2-1, \ldots, |2J_1-2J_2|$. We now consider the irrep J and its subspace with degree $\lambda(2J)$. Because $2J-M$ is even, if $2(J_1+J_2-J)$ is even, then $2J_1-M_1$ and $2J_2-M_2$ are all even or odd; if $2(J_1+J_2-J)$ is odd, then $2J_1-M_1$ is even, $2J_2-M_2$ odd, or $2J_1-M_1$ is odd and $2J_2-M_2$ even. From $\lambda(M) = \lambda(M_1)+\lambda(M_2)$ (see (3.2) below), we obtain $\lambda(2J) = \lambda(2J_1)+\lambda(2J_2)$ for even $2(J_1+J_2-J)$ and $\lambda(2J) = \lambda(2J_1)+\lambda(2J_2)+1$ for odd $2(J_1+J_2-J)$. The two cases can be combined into a single formula (with a difference of an even integer):

$$\lambda(2J) = \lambda(2J_1)+\lambda(2J_2)+2(J_1+J_2-J). \tag{3.1}$$

One sees from (2.6) and (3.1) that the relation $\varepsilon(2J, M)=\varepsilon(2J, M+1)$ cannot be obtained simultaneously in the two cases $2J\in\{2J_1+2J_2,\ 2J_1+2J_2-2,\dots\}$ and $2J\in\{2J_1+2J_2-1,\ 2J_1+2J_2-3,\dots\}$. Therefore, the metrics of the coupling spaces are positive for some subspaces and negative for other subspaces, regardless of whether the metrics of the in-coupling spaces are positive or negative. Thus, if we want to study the coupling generally, for the cases of the coupling of three or four irreps in particular, it is necessary to introduce an indefinite metric.

3.2. CG coefficients

We denote the basis vectors of the in-coupling spaces J_1 and J_2 by $|2J_1, M_1\rangle$ and $|2J_2, M_2\rangle$, respectively, and the basis vectors of the coupling spaces by $|2J, M\rangle$. We express the relations between the basis vectors of the coupling spaces and those of the in-coupling spaces as

$$|2J, M\rangle=\sum_{M_1M_2}\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2 & M\end{pmatrix}|2J_1, M_1\rangle|2J_2, M_2\rangle \tag{3.2}$$

$$\varepsilon(2J_2, M_2)|2J_2, M_2\rangle\varepsilon(2J_1, M_1)|2J_1, M_1\rangle=\sum_{2J}\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2 & M\end{pmatrix}\varepsilon(2J, M)|2J, M\rangle \tag{3.3}$$

where ( ) is the CG coefficient, $M=M_1+M_2$.

From (3.2) and (3.3), we can derive the orthogonality conditions of the CG coefficients:

$$\sum_{M_1M_2}(-1)^{\lambda(M_1)\lambda(M_2)}\varepsilon(2J_1, M_1)\varepsilon(2J_2, M_2)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2 & M\end{pmatrix}\begin{pmatrix}2J_1 & 2J_2 & 2J'\\ M_1 & M_2 & M'\end{pmatrix}=\varepsilon(2J, M)\delta_{J'J}\delta_{M'M} \tag{3.4}$$

$$\sum_{2J}\varepsilon(2J, M)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2 & M\end{pmatrix}\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1' & M_2' & M\end{pmatrix}=(-1)^{\lambda(M_1)\lambda(M_2)}\varepsilon(2J_1, M_1)\varepsilon(2J_2, M_2)\delta_{M_1'M_1}\delta_{M_2'M_2} \tag{3.5}$$

and their recursion relations

$$A_\pm(M)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2 & M\pm2\end{pmatrix}=A_\pm(M_1\mp2)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1\mp2 & M_2 & M\end{pmatrix}+A_\pm(M_2\mp2)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2\mp2 & M\end{pmatrix} \tag{3.6}$$

$$B_\pm(M)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2 & M\pm1\end{pmatrix}=B_\pm(M_1\mp1)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1\mp1 & M_2 & M\end{pmatrix}+(-1)^{\lambda(M_1)}B_\pm(M_2\mp1)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1 & M_2\mp1 & M\end{pmatrix}. \tag{3.7}$$

Eight kinds of CG coefficients exist, which are classified according to the odd–even properties of $2J_1-M_1$, $2J_2-M_2$, $2(J_1+J_2-J)$ and $2J-M$ (see table 1). By means of the relations (3.4)–(3.7) and a convenient choice of the relative phase between the CG coefficients with even $2(J_1+J_2-J)$ and those with odd $2(J_1+J_2-J)$, we can obtain eight kinds of CG coefficients (which are determined up to a common sign) as follows:

$$\begin{aligned}
&1 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = \left(\frac{J_1+J_2+J+1}{2J+1}\right)^{1/2} C^{2J_1\ 2J_2\ 2J}_{M_1\ M_2\ M} \\
&2 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = \left(\frac{J_1+J_2+J+\frac{1}{2}}{2J}\right)^{1/2} C^{2J_1-1\ 2J_2-1\ 2J-1}_{M_1\ M_2\ M} \\
&3 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = (-1)^{\lambda(2J_1)+1}\left(\frac{J_1+J_2-J+\frac{1}{2}}{2J}\right)^{1/2} C^{2J_1\ 2J_2\ 2J-1}_{M_1\ M_2\ M} \\
&4 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = (-1)^{\lambda(2J_1)+1}\left(\frac{J_1+J_2-J}{2J+1}\right)^{1/2} C^{2J_1-1\ 2J_2-1\ 2J}_{M_1\ M_2\ M} \\
&5 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = (-1)^{\lambda(2J_1)}\left(\frac{J-J_1+J_2}{2J}\right)^{1/2} C^{2J_1\ 2J_2-1\ 2J-1}_{M_1\ M_2\ M} \\
&6 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = \left(\frac{J_1-J_2+J+\frac{1}{2}}{2J+1}\right)^{1/2} C^{2J_1\ 2J_2-1\ 2J}_{M_1\ M_2\ M} \\
&7 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = \left(\frac{J_1-J_2+J}{2J}\right)^{1/2} C^{2J_1-1\ 2J_2\ 2J-1}_{M_1\ M_2\ M} \\
&8 \quad \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = (-1)^{\lambda(2J_1)+1}\left(\frac{J-J_1+J_2+\frac{1}{2}}{2J+1}\right)^{1/2} C^{2J_1-1\ 2J_2\ 2J}_{M_1\ M_2\ M}
\end{aligned} \tag{3.8}$$

where C on the right of (3.8) are the CG coefficients of the SO(3) algebra, its connection with the usual form $C^{J_1\ J_2\ J}_{m_1\ m_2\ m}$ being

$$C^{2J_1\ 2J_2\ 2J}_{M_1\ M_2\ M} = C^{J_1\ J_2\ J}_{m_1\ m_2\ m} \tag{3.9}$$

where $m=\frac{1}{2}M$, $m_i=\frac{1}{2}M_i$.

The proportional factors in (3.8) are actually the isoscalar factors, and their values are as the same as those given by Scheunert *et al* [2].

Table 1. CG coefficients (+, even; −, odd).

	$2J_1-M_1$	$2J_2-M_2$	$2(J_1+J_2-J)$	$2J-M$
1	+	+	+	+
2	−	−	−	−
3	+	+	−	−
4	−	−	+	+
5	+	−	+	−
6	+	−	−	+
7	−	+	+	−
8	−	+	−	+

Through the analysis of the symmetry properties included in (3.8), one can see that the CG coefficients of the OSP(1, 2) algebra satisfy the following symmetry relations:

$$\begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1 & M_2 & M \end{pmatrix} = (-1)^{f_1}\begin{pmatrix} 2J_2 & 2J_1 & 2J \\ M_2 & M_1 & M \end{pmatrix} \tag{3.10}$$

$$= (-1)^{f_2}\begin{pmatrix} 2J_1 & 2J_2 & 2J \\ -M_1 & -M_2 & -M \end{pmatrix} \tag{3.11}$$

$$= (-1)^{f_3}\begin{pmatrix} 2J_1 & 2J & 2J_2 \\ M_1 & -M & -M_2 \end{pmatrix} \tag{3.12}$$

$$= (-1)^{f_4}\begin{pmatrix} 2J & 2J_2 & 2J_1 \\ -M & M_2 & -M_1 \end{pmatrix} \tag{3.13}$$

where

$$f_1 = n(J_1J_2J) + 2(J_1+J_2-J)\lambda(2J) + \lambda(2J_1)\lambda(2J_2) + \lambda(M_1)\lambda(M_2)$$

$$f_2 = n(J_1J_2J) + 2(J_1+J_2-J)(2J-M) + (2J_1-M_1)(2J_2-M_2)$$

$$f_3 = \tfrac{1}{2}(M_1^{>} - M_1) + [\lambda(2J_1)+1][2(J_1+J_2-J)+\lambda(M_1)]$$

$$f_4 = \tfrac{1}{2}(M_2^{>} + M_2) + [\lambda(2J_2)+1][2(J_1+J_2-J)+\lambda(M_2)]$$

while $n(J_1J_2J)$ equals J_1+J_2-J for even $2(J_1+J_2-J)$ and $J_1+J_2-J-\frac{1}{2}$ for odd $2(J_1+J_2-J)$, and $M_i^{>}$ equals $2J_i$ for even $(2J_i-M_i)$ and $2J_i-1$ for odd $(2J_i-M_i)$.

3.3. Metrics of coupling spaces

We may show by means of the orthogonality conditions and the formulae for calculating CG coefficients that there is a relation between the metrics of the coupling spaces and that of the in-coupling spaces, namely

$$\varepsilon(2J, M) = (-1)^{2(J_1+J_2-J)(2J-M)+(2J_1-M_1)(2J_2-M_2)+\lambda(M_1)\lambda(M_2)}\varepsilon(2J_1, M_1)\varepsilon(2J_2, M_2). \tag{3.14}$$

4. Racah coefficients of the OSP(1, 2) algebra

4.1. Definition of the Racah coefficient

In order to study the coupling of three irreps, we may follow the method used in SO(3) algebra [3]. There are two coupling approaches; in the first J_1, J_2 are first coupled to J' and J', J_3 then are coupled to J. In the second, J_2, J_3 are first coupled to J'' and J_1, J'' then are coupled to J. We denote the final state vectors by $|2J, M\rangle_{J'}$ and $|2J, M\rangle_{J''}$ which correspond to the middle states J' and J'' respectively, and write the relations between the final state vectors of the two kinds as

$$|2J, M\rangle_{J'} = \sum_{J''} R_{J''J'}|2J, M\rangle_{J''} \tag{4.1}$$

$$\varepsilon(2J, M)_{J''}|2J, M\rangle_{J''} = \sum_{J'} R_{J''J'}\varepsilon(2J, M)_{J'}|2J, M\rangle_{J'} \tag{4.2}$$

where $R_{J''J'} = R(J_1J_2JJ_3; J'J'')$ is the Racah coefficient which is assumed to be real. Because M on the two sides of (4.1) and (4.2) are the same, R does not depend on

M. By making use of (3.2), the final state vectors of the two kinds can be expressed in terms of the basis vectors of the in-coupling spaces:

$$|2J, M\rangle_{J'} = \sum_{\substack{M_1 M_2 \\ M_3 M'}} \begin{pmatrix} 2J_1 & 2J_2 & 2J' \\ M_1 & M_2 & M' \end{pmatrix} \times \begin{pmatrix} 2J' & 2J_3 & 2J \\ M' & M_3 & M \end{pmatrix} |2J_1, M_1\rangle |2J_2, M_2\rangle |2J_3, M_3\rangle \tag{4.3}$$

$$|2J, M\rangle_{J''} = \sum_{\substack{M_1 M_2 \\ M_3 M''}} \begin{pmatrix} 2J_2 & 2J_3 & 2J'' \\ M_2 & M_3 & M'' \end{pmatrix} \times \begin{pmatrix} 2J_1 & 2J'' & 2J \\ M_1 & M'' & M \end{pmatrix} |2J_1, M_1\rangle |2J_2, M_2\rangle |2J_3, M_3\rangle. \tag{4.4}$$

Substituting the above expressions into (4.1), we obtain

$$\begin{pmatrix} 2J_1 & 2J_2 & 2J' \\ M_1 & M_2 & M' \end{pmatrix} \begin{pmatrix} 2J' & 2J_3 & 2J \\ M' & M_3 & M \end{pmatrix} = \sum_{J''} R_{J''J'} \begin{pmatrix} 2J_2 & 2J_3 & 2J'' \\ M_2 & M_3 & M'' \end{pmatrix} \begin{pmatrix} 2J_1 & 2J'' & 2J \\ M_1 & M'' & M \end{pmatrix}. \tag{4.5}$$

Using the orthogonality conditions (3.4) of the CG coefficients, we can obtain from (4.5)

$$R_{J''J'} \begin{pmatrix} 2J_1 & 2J'' & 2J \\ M_1 & M'' & M \end{pmatrix} = \sum_{M_2 M_3 M'} (-1)^{\lambda(M_2)\lambda(M_3)} \varepsilon(2J_2, M_2) \varepsilon(2J_3, M_3) \varepsilon(2J'', M'') \times \begin{pmatrix} 2J_1 & 2J_2 & 2J' \\ M_1 & M_2 & M' \end{pmatrix} \begin{pmatrix} 2J' & 2J_3 & 2J \\ M' & M_3 & M \end{pmatrix} \begin{pmatrix} 2J_2 & 2J_3 & 2J'' \\ M_2 & M_3 & M'' \end{pmatrix} \tag{4.6}$$

and

$$R_{J''J'} = \sum_{\substack{M_1 M_2 M_3 \\ M' M''}} P(M_1 M_2 M_3) \begin{pmatrix} 2J_1 & 2J_2 & 2J' \\ M_1 & M_2 & M' \end{pmatrix} \begin{pmatrix} 2J' & 2J_3 & 2J \\ M' & M_3 & M \end{pmatrix} \times \begin{pmatrix} 2J_2 & 2J_3 & 2J'' \\ M_2 & M_3 & M'' \end{pmatrix} \begin{pmatrix} 2J_1 & 2J'' & 2J \\ M_1 & M'' & M \end{pmatrix} \tag{4.7}$$

where

$$P(M_1 M_2 M_3) = (-1)^{\lambda(M_1)\lambda(M_2) + \lambda(M_2)\lambda(M_3) + \lambda(M_3)\lambda(M_1)} \times \varepsilon(2J_1, M_1) \varepsilon(2J_2, M_2) \varepsilon(2J_3, M_3) \varepsilon(2J, M)_{J''}.$$

4.2. Calculating formulae for Racah coefficients

Now we consider $\lambda(2J)$. In the coupling approach where $J_1, J_2 \to J'$; $J', J_3 \to J$, we have $\lambda(2J) = \lambda(2J_1) + \lambda(2J_2) + \lambda(2J_3) + 2(J_1 + J_2 - J') + 2(J' + J_3 - J)$; in the coupling approach where $J_2, J_3 \to J''$; $J_1, J'' \to J$, we have $\lambda(2J) = \lambda(2J_1) + \lambda(2J_2) + \lambda(2J_3) + 2(J_2 + J_3 - J'') + 2(J_1 + J'' - J)$. Because $\lambda(2J)$ is the same for the two coupling approaches, we obtain (with a difference of an even integer)

$$2(J_1 + J_2 - J') + 2(J' + J_3 - J) = 2(J_2 + J_3 - J'') + 2(J_1 + J'' - J). \tag{4.8}$$

We find from (4.8) that eight kinds of Racah coefficients exist which are classified according to the odd–even properties of $2(J_1+J_2-J')$, $2(J'+J_3-J)$, $2(J_2+J_3-J'')$ and $2(J_1+J''-J)$ (see table 2).

A simpler way of calculating Racah coefficients is to use (4.6). In order to derive a set of calculating formulae, we must first make a convenient choice of the odd–even properties of $2J_1-M_1$ and $2J''-M''$, then move the CG coefficient of the SO(3) algebra from the left-hand side of (4.6) to the right-hand side, introduce the Racah coefficient of the SO(3) algebra, and use (3.14) to eliminate the metric factors in (4.6). The calculating formulae are the following ($2J_1-M_1$ and $2J''-M''$ have been chosen to be even):

1 $R(J_1J_2JJ_3;\, J'J'')$

$$= (J_1+J''+J+1)^{-1/2}\{[(J_1+J_2+J'+1)(J'+J_3+J+1)(J_2+J_3+J''+1)]^{1/2} \times W(J_1J_2JJ_3;\, J'J'') - [(J'-J_1+J_2)(J'+J_3-J)(J_2+J_3-J'')]^{1/2} \times W(J_1J_2-\tfrac{1}{2}, JJ_3-\tfrac{1}{2};\, J'-\tfrac{1}{2}, J'')\} \tag{4.9}$$

2 $R(J_1J_2JJ_3;\, J'J'')$

$$= (J_1+J''-J+\tfrac{1}{2})^{-1/2}\{[(J_1+J_2-J'+\tfrac{1}{2})(J'+J_3+J+\tfrac{1}{2})(J_2-J_3+J''+\tfrac{1}{2})]^{1/2} \times W(J_1J_2J-\tfrac{1}{2}, J_3-\tfrac{1}{2};\, J'-\tfrac{1}{2}, J'') + [(J_1-J_2+J'+\tfrac{1}{2})(J'+J_3-J+\tfrac{1}{2})(J''-J_2+J_3+\tfrac{1}{2})]^{1/2} \times W(J_1J_2-\tfrac{1}{2}, J-\tfrac{1}{2}, J_3;\, J'J'')\} \tag{4.10}$$

3 $R(J_1J_2JJ_3;\, J'J'')$

$$= (-1)^{\lambda(2J_2)+1}(J_1+J''-J+\tfrac{1}{2})^{-1/2} \times\{[(J_1+J_2+J'+1)(J-J'+J_3)(J_2-J_3+J''+\tfrac{1}{2})]^{1/2} \times W(J_1J_2J-\tfrac{1}{2}, J_3-\tfrac{1}{2};\, J'J'') - [(J'-J_1+J_2)(J'-J_3+J)(J''-J_2+J_3+\tfrac{1}{2})]^{1/2} \times W(J_1J_2-\tfrac{1}{2}, J-\tfrac{1}{2}, J_3;\, J'-\tfrac{1}{2}, J'')\} \tag{4.11}$$

4 $R(J_1J_2JJ_3;\, J'J'')$

$$= (-1)^{\lambda(2J_2)+1}(J_1+J''+J+1)^{-1/2} \times\{[(J_1+J_2-J'+\tfrac{1}{2})(J-J'+J_3+\tfrac{1}{2})(J_2+J_3+J''+1)]^{1/2} \times W(J_1J_2JJ_3;\, J'-\tfrac{1}{2}, J'') - [(J_1-J_2+J'+\tfrac{1}{2})(J'-J_3+J+\tfrac{1}{2})(J_2+J_3-J'')]^{1/2} \times W(J_1J_2-\tfrac{1}{2}, JJ_3-\tfrac{1}{2};\, J'J'')\} \tag{4.12}$$

Table 2. Racah coefficients (+, even; −, odd).

	$2(J_1+J_2-J')$	$2(J'+J_3-J)$	$2(J_2+J_3-J'')$	$2(J_1+J''-J)$
1	+	+	+	+
2	−	−	−	−
3	+	+	−	−
4	−	−	+	+
5	+	−	+	−
6	+	−	−	+
7	−	+	+	−
8	−	+	−	+

5 $R(J_1J_2JJ_3;\ J'J'')$

$$
\begin{aligned}
&=(-1)^{\lambda(2J_2)}(J_1+J''-J+\tfrac{1}{2})^{-1/2}\\
&\quad\times\{[(J_1+J_2+J'+1)(J'+J_3-J+\tfrac{1}{2})(J_2+J_3+J''+1)]^{1/2}\\
&\quad\times W(J_1J_2J-\tfrac{1}{2},J_3;\ J'J'')-[(J'-J_1+J_2)(J'+J_3+J+\tfrac{1}{2})(J_2+J_3-J'')]^{1/2}\\
&\quad\times W(J_1J_2-\tfrac{1}{2},J-\tfrac{1}{2},J_3-\tfrac{1}{2};\ J'-\tfrac{1}{2},J'')\}
\end{aligned}
\tag{4.13}
$$

6 $R(J_1J_2JJ_3;\ J'J'')$

$$
\begin{aligned}
&=(J_1+J''+J+1)^{-1/2}\{[(J_1+J_2+J'+1)(J'-J_3+J+\tfrac{1}{2})(J_2-J_3+J''+\tfrac{1}{2})]^{1/2}\\
&\quad\times W(J_1J_2JJ_3-\tfrac{1}{2};\ J'J'')\\
&\quad+[(J'-J_1+J_2)(J-J'+J_3+\tfrac{1}{2})(J''-J_2+J_3+\tfrac{1}{2})]^{1/2}\\
&\quad\times W(J_1J_2-\tfrac{1}{2},JJ_3;\ J'-\tfrac{1}{2},J'')\}
\end{aligned}
\tag{4.14}
$$

7 $R(J_1J_2JJ_3;\ J'J'')$

$$
\begin{aligned}
&=(J_1+J''-J+\tfrac{1}{2})^{-1/2}\{[(J_1+J_2-J'+\tfrac{1}{2})(J'-J_3+J)(J_2+J_3+J''+1)]^{1/2}\\
&\quad\times W(J_1J_2J-\tfrac{1}{2},J_3;\ J'-\tfrac{1}{2},J'')\\
&\quad+[(J_1-J_2+J'+\tfrac{1}{2})(J-J'+J_3)(J_2+J_3-J'')]^{1/2}\\
&\quad\times W(J_1J_2-\tfrac{1}{2},J-\tfrac{1}{2},J_3-\tfrac{1}{2};\ J'J'')\}
\end{aligned}
\tag{4.15}
$$

8 $R(J_1J_2JJ_3;\ J'J'')$

$$
\begin{aligned}
&=(-1)^{\lambda(2J_2)+1}(J_1+J''+J+1)^{-1/2}\\
&\quad\times\{[(J_1+J_2-J'+\tfrac{1}{2})(J'+J_3-J)(J_2-J_3+J''+\tfrac{1}{2})]^{1/2}\\
&\quad\times W(J_1J_2JJ_3-\tfrac{1}{2};\ J'-\tfrac{1}{2},J'')\\
&\quad+[(J_1-J_2+J'+\tfrac{1}{2})(J'+J_3+J+1)(J''-J_2+J_3+\tfrac{1}{2})]^{1/2}\\
&\quad\times W(J_1J_2-\tfrac{1}{2},JJ_3;\ J'J'')\}
\end{aligned}
\tag{4.16}
$$

where $W(\ )$ are the Racah coefficients of the SO(3) algebra [3].

The simpler Racah coefficients corresponding to the $J'=0$ and $\frac{1}{2}$ cases are shown in the appendix.

4.3. *Orthogonality conditions of Racah coefficients*

From the definitions (4.1) and (4.2) for R, we can easily derive the orthogonality conditions of the Racah coefficients:

$$\sum_{J''}\varepsilon(2J,M)_{J''}R_{J''J'}R_{J''J'}=\varepsilon(2J,M)_{J'}\delta_{J'J'} \tag{4.17}$$

$$\sum_{J'}\varepsilon(2J,M)_{J'}R_{J''J'}R_{J''J'}=\varepsilon(2J,M)_{J''}\delta_{J''J''}. \tag{4.18}$$

It is noteworthy that the metrics of the final states with different middle states are generally different by a phase factor:

$$\varepsilon(2J, M)_{J'} = (-1)^{2(J_1+J_2-J')2(J'+J_3-J)+2(J_2+J_3-J'')2(J_1+J''-J)}\varepsilon(2J, M)_{J''}. \tag{4.19}$$

We can prove (4.19) by means of (3.14) and (4.8).

Substituting the relation between $\varepsilon(2J, M)_{J'}$ and $\varepsilon(2J, M)_{J''}$ into (4.17) and (4.18), we obtain

$$\sum_{J''} (-1)^{2(J_2+J_3-J'')2(J_1+J''-J)} R_{J''J'} R_{J''J'} = (-1)^{2(J_1+J_2-J')2(J'+J_3-J)} \delta_{J'J'} \tag{4.20}$$

$$\sum_{J'} (-1)^{2(J_1+J_2-J')2(J'+J_3-J)} R_{J''J'} R_{J''J'} = (-1)^{2(J_2+J_3-J'')2(J_1+J''-J)} \delta_{J''J''}. \tag{4.21}$$

4.4. *Symmetry properties of Racah coefficients*

The Racah coefficients of the OSP(1, 2) algebra satisfy the following symmetry relations (for simplicity, we replace $J_1J_2JJ_3J'J''$ by *abcdef* respectively):

$$(-1)^{\lambda(2a)2(a+d+e+f)+2(a+b+e)2(b+d)} R(badc;\ ef) = (-1)^{\lambda(2b)2(b+c+e+f)+2(a+b+e)2(a+c)} R(abcd;\ ef) \tag{4.22}$$

$$(-1)^{\lambda(2d)2(a+d+e+f)+2(b+d+f)2(c+d)} R(cdab;\ ef) = (-1)^{\lambda(2b)2(b+c+e+f)+2(b+d+f)2(a+b)} R(abcd;\ ef) \tag{4.23}$$

$$(-1)^{\lambda(2c)2(b+c+e+f)+2(c+f)} R(acbd; fe) = (-1)^{\lambda(2b)2(b+c+e+f)+2(b+e)} R(abcd; ef) \tag{4.24}$$

$$(-1)^{n(afc)+2(a+f+c)[\lambda(2a)+2(b+d+f)+1]} R(aefd;\ bc) = (-1)^{n(abe)+2(a+b+e)[\lambda(2a)+2(e+d+c)+1]} R(abcd;\ ef). \tag{4.25}$$

These symmetry relations can be proved by means of the symmetry relations of CG coefficients. The other symmetry relations could be obtained from (4.22)-(4.25). The reader may check the symmetry relations one by one using the examples in the appendix.

5. Conclusion

OSP(1, 2) is one of the few Lie superalgebras for which coupling coefficients may be introduced. We have calculated the coupling coefficients of the OSP(1, 2) algebra and determined their properties. There are eight kinds of CG coefficients and eight kinds of Racah coefficients of OSP(1, 2) algebra and they are all expressed in terms of the corresponding coefficients of the SO(3) algebra. These results strongly reflect the connection between the representations of the OSP(1, 2) and SO(3) algebras.

Appendix. A table of Racah coefficients

We give the Racah coefficients for $J'=0, \frac{1}{2}$. For simplicity, the letters $J_1\ J_2\ J\ J_3\ J'\ J''$ are replaced by *a b c d e f*, respectively.

(*i*) $e=0$

1 $\quad R(bbdd;\ 0f) = (-1)^{b+d-f}$

3 $R(bbdd;\, 0f)=(-1)^{\lambda(2b)+b+d-f+1/2}.$

(*ii*) $e=\frac{1}{2}$

1

a	c: $d+\frac{1}{2}$	c: $d-\frac{1}{2}$
$b+\frac{1}{2}$	$(-1)^{b+d-f}\times\left(\frac{(b+d+f+1)(b+d-f+1)}{(2b+1)(2d+1)}\right)^{1/2}$	$(-1)^{b+d-f}\times\left(\frac{(b-d+f+1)(-b+d+f)}{(2b+1)2d}\right)^{1/2}$
$b-\frac{1}{2}$	$(-1)^{b+d-f}\times\left(\frac{(-b+d+f+1)(b-d+f)}{2b(2d+1)}\right)^{1/2}$	$(-1)^{b+d-f+1}\times\left(\frac{(b+d+f)(b+d-f)}{2b2d}\right)^{1/2}$

2 $$R(bbdd;\, \tfrac{1}{2}f)=(-1)^{b+d-f-1/2}\frac{4bd+b+d+f+\frac{1}{2}}{[2b(2b+1)2d(2d+1)]^{1/2}}$$

3

a	c: $d+\frac{1}{2}$	c: $d-\frac{1}{2}$
$b+\frac{1}{2}$	$(-1)^{\lambda(2b)+b+d-f+1/2}\times\left(\frac{(b+d-f+\frac{1}{2})(b+d+f+\frac{3}{2})}{(2b+1)(2d+1)}\right)^{1/2}$	$(-1)^{\lambda(2b)+b+d-f+1/2}\times\left(\frac{(b-d+f+\frac{1}{2})(-b+d+f-\frac{1}{2})}{(2b+1)2d}\right)^{1/2}$
$b-\frac{1}{2}$	$(-1)^{\lambda(2b)+b+d-f+1/2}\times\left(\frac{(-b+d+f+\frac{1}{2})(b-d+f-\frac{1}{2})}{2b(2d+1)}\right)^{1/2}$	$(-1)^{\lambda(2b)+b+d-f-1/2}\times\left(\frac{(b+d-f-\frac{1}{2})(b+d+f+\frac{1}{2})}{2b2d}\right)^{1/2}$

4 $$R(bbdd;\, \tfrac{1}{2}f)=(-1)^{\lambda(2b)+b+d-f+1}\frac{4bd+b+d-f}{[2b(2b+1)2d(2d+1)]^{1/2}}$$

5

a	c: d
$b+\frac{1}{2}$	$(-1)^{\lambda(2b)+b+d-f}\left(\frac{(b+d+f+1)(-b+d+f)}{(2b+1)2d(2d+1)}\right)^{1/2}$
$b-\frac{1}{2}$	$(-1)^{\lambda(2b)+b+d-f+1}\left(\frac{(b-d+f)(b+d-f)}{2b2d(2d+1)}\right)^{1/2}$

6

a	c: d
$b+\frac{1}{2}$	$(-1)^{b+d-f-1/2}\left(\frac{(b-d+f+\frac{1}{2})(b+d-f+\frac{1}{2})}{(2b+1)2d(2d+1)}\right)^{1/2}$
$b-\frac{1}{2}$	$(-1)^{b+d-f-1/2}\left(\frac{(b+d+f+\frac{1}{2})(-b+d+f+\frac{1}{2})}{2b2d(2d+1)}\right)^{1/2}$

7

a	c: $d+\frac{1}{2}$	c: $d-\frac{1}{2}$
b	$(-1)^{b+d-f} \times \left(\frac{(b+d+f+1)(b-d+f)}{2b(2b+1)(2d+1)}\right)^{1/2}$	$(-1)^{b+d-f+1} \times \left(\frac{(b+d-f)(-b+d+f)}{2b(2b+1)2d}\right)^{1/2}$

8

a	c: $d+\frac{1}{2}$	c: $d-\frac{1}{2}$
b	$(-1)^{\lambda(2b)+b+d-f+1/2} \times \left(\frac{(-b+d+f+\frac{1}{2})(b+d-f+\frac{1}{2})}{2b(2b+1)(2d+1)}\right)^{1/2}$	$(-1)^{\lambda(2b)+b+d-f+1/2} \times \left(\frac{(b+d+f+\frac{1}{2})(b-d+f+\frac{1}{2})}{2b(2b+1)2d}\right)^{1/2}$

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